Série 5a Solutions

Exercise 5a.1 – Torsion of a spinning ship

The ship at A has just started to drill for oil on the ocean floor at a depth of 1525 m (Figure 5a.1). Knowing that the top of the 0.2 m diameter steel drill pipe ($G = 77.2$ GPa) rotates through two complete revolutions before the drill bit at B starts to operate, **determine the maximum shearing stress caused in the pipe by torsion.**

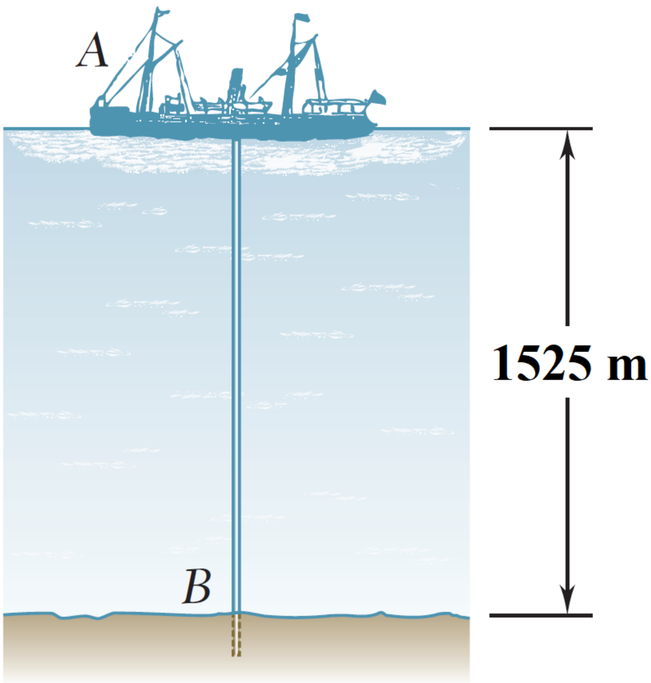


Figure 5a.1 | Drilling ship in torsion.

Solution – Exercise 5a.1

Find the maximum shearing stress caused in the pipe by torsion

$$\boldsymbol{L = 1525\ \text{m}}$$

$$c = \frac{1}{2}d = 0.1\ \text{m} \quad (5a.1.1)$$

$$\phi = \frac{TL}{GI_P} \rightarrow T = \frac{GI_P\phi}{L} \quad (5a.1.2)$$

$$\tau = \frac{Tc}{I_P} = \frac{GI_P\phi}{L} * \left(\frac{c}{I_P}\right) = \frac{G\phi c}{L} \quad (5a.1.3)$$

$$\phi = 2\ \text{rev} = 2 * 2\pi = 12.566\ \text{rad} \quad (5a.1.4)$$

$$\tau = \frac{(77.2 \cdot 10^9\ \text{Pa})(12.566\ \text{rad})(0.1\ \text{m})}{1525\ \text{m}} = 63614622\ \text{Pa} \approx 63.6\ \text{MPa} \quad (5a.1.5)$$

Exercise 5a.2 – Torque of a composite bar in torsion

Consider the following composite bar in Figure 5a.2 (the shear moduli are indicated in parenthesis). The cross-section is circular. The bar is clamped at one end. A torsion angle of 2° is measured at the free end of the bar.

Determine the torque, *T*, applied at the free end of the bar.

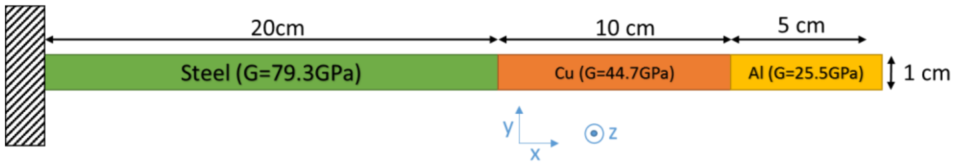


Figure 5a.2 | Composite bar.

Solution – 5a.2

Determine the torque, *T*, applied at the free end of the bar

First consider the polar moment of inertia I_p. It only depends on the geometry of the bar. We call d the diameter of the bar.

$$I_p = \frac{\pi}{32} d^4 \tag{5a.2.1}$$

Then, we calculate the separate stiffness k of the three respective portions of the bar. Calling L the length of the bar, it gives us for each portion:

$$k_x = G_x \frac{I_p}{L_x} \tag{5a.2.2}$$

The equivalent stiffness of serial portions is the indirect sum of the respective stiffness of each portion of the bar. Therefore:

$$k_{eq} = \left(k_{green}^{-1} + k_{orange}^{-1} + k_{yellow}^{-1}\right)^{-1} \tag{5a.2.3}$$

To conclude, we obtain the torsion T with respect to the torsion angle ϕ thanks to the law:

$$T = k_{eq} * \phi \tag{5a.2.4}$$

Substituting, the numerical application gives:

$$T = 5.1 \text{ N} \cdot \text{m} \tag{5a.2.5}$$

Exercise 5a.3 – Torsion along a bar

A bar AB of solid cross section (diameter d) is loaded by a distributed torque (see Figure 5a.3). The intensity of the torque, the torque per unit distance, is denoted *t(x)* and varies linearly from a maximum value t_A at the end A to zero at the end B. The length of the bar is L and the shear modulus of the material is G.

- **a) Calculate the internal torque in the bar as a function of x, $T_{int}(x)$.**
- **b) Determine the point of maximum internal torque in the bar.**
- **c) Determine the maximum internal torque, T_{max}, and the maximum shear stress, τ_{max}, in the bar.**
- **d) Determine the angle of twist, ϕ, between the ends of the bar.**

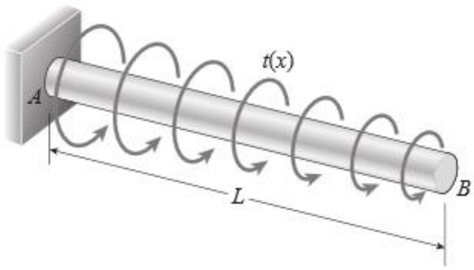


Figure 5a.3 | Bar under torsion.

Solution 5a.3

<u>What is given?</u>

Torque per unit distance in A: t_A

Torque per unit distance in B: 0

Shear modulus: G

Length of the bar: L

Diameter of the bar: d

<u>Principles</u>

The torque per unit distance varies linearly from point A to B.

The shear stress distribution when the Shear modulus is not constant.

<u>What is asked</u>

Calculate the torque in the bar as a function of x
Determine the point of maximum torque in the bar.
Determine the maximum torque T_{max} and the maximum shear-stress τ_{max} in the bar.
Determine the angle of twist ϕ between the ends of the bar.

<u>Equations required</u>

Angle of twist:

$$\phi = \int \left(\frac{T(x)}{GI_p}\right) dx \qquad (5a.3.1)$$

Second moment of inertia for circle cross-section

$$I_p = \frac{\pi d^4}{32} \quad (5a.3.2)$$

General torsion formula:

$$T = \int_A \tau(r) r dA \quad (5a.3.3)$$

Torsion formula for circular cross-section body:

$$T = \frac{2\tau_{max} I_p}{d} \quad (5a.3.4)$$

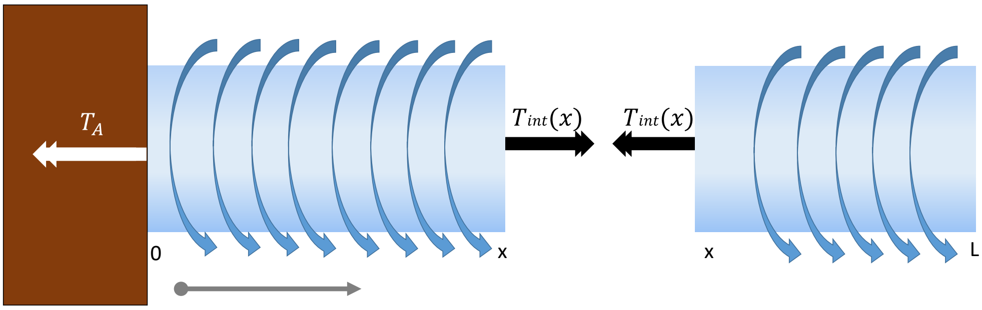


Figure 5a.3.2| Schematic showing the differential section taken to integrate the moment.

Solution approach OPTION A

We place the origin in point A and positive pointing towards B.

a) Calculate the internal torque in the bar as a function of x, $T_{int}(x)$.

If we cut a section of the beam and **we take the section from x to L (point B),** we can write:

$$\int_x^L t(x')\,dx' - T_{int} = 0 \rightarrow T_{int}(x) = \int_x^L t(x')\,dx' = \int_x^L t_a\left(1 - \frac{x'}{L}\right)dx' \quad (5a.3.5)$$

$$T_{int}(x) = t_a\left(L - x - \frac{L^2 - x^2}{2L}\right) = \frac{t_a L}{2}\left(1 - \frac{x}{L}\right)^2 \quad (5a.3.6)$$

b) Determine the point of maximum internal torque in the bar.

We can clearly see that $T_{int}(x)$ is always positive with a maximum value for $x = 0$, which means at point A:

$$x_{max} = 0 \quad (5a.3.7)$$

c) Determine the maximum internal torque, T_{max}, and the maximum shear stress, τ_{max}, in the bar

From (b), the maximum internal torque is then:

$$T_{max} = T_{int}(0) = \frac{L t_a}{2} \quad (5a.3.8)$$

The maximum shear-stress is gotten from Torque formula:

$$T_{int} = 2\frac{\tau_{max} I_p}{d} \tag{5a.3.9}$$

This implies

$$\tau_{max} = \left(\frac{\pi\left(\frac{d}{2}\right)^3}{2T}\right)^{-1} = \frac{2T_{max}}{\pi\left(\frac{d}{2}\right)^3} = \frac{8Lt_a}{\pi d^3} \tag{5a.3.10}$$

d) Determine the angle of twist, ϕ, between the ends of the bar

Finally, to calculate the angle we use:

$$d\phi = \frac{T_{int}(x)}{GI_p}dx \rightarrow \phi(x=L) - \phi(0) = \int_0^L \frac{T_{int}(x)}{GI_p}dx \rightarrow$$
$$\phi_B = \phi(x=L) = \int_0^L \frac{T_{int}(x)}{GI_p}dx \tag{5a.3.11}$$

Then:

$$\phi_B = \int_0^L \frac{t_a L}{2GI_p}\left(1-\frac{x}{L}\right)^2 dx = \frac{t_a L}{2GI_p}\left[-\frac{L}{3}\left(1-\frac{x}{L}\right)^3\right]_0^L = \frac{t_a}{GI_p}\frac{L^2}{6} = \frac{16}{3\pi}\frac{L^2 t_a}{Gd^4} \tag{5a.3.12}$$

Solution approach OPTION B

We place the origin in point A and positive pointing towards B.

a) Calculate the internal torque in the bar as a function of x, $T_{int}(x)$.

If we cut a section of the beam and **we take the section from 0 (point A) to x,** we can write:

$$\int_0^x t(x')\,dx' + T_{int}(x) - T_A = 0 \rightarrow T_{int}(x) = T_A - \int_0^x t(x')\,dx' \tag{5a.3.13}$$

Where T_A is the reaction torque required to keep equilibrium in the bar:

$$T_A = \int_0^L t(x')\,dx' \tag{5a.3.14}$$

So using Eq 5a.3.13 in Eq 5a.3.14, we have:

$$T_{int}(x) = T_A - \int_0^x t(x')\,dx' = \int_0^L t(x')\,dx' - \int_0^x t(x')\,dx'$$
$$= \int_x^L t(x')\,dx' = \int_x^L t_a\left(1-\frac{x'}{L}\right)dx' \tag{5a.3.15}$$

$$T_{int}(x) = t_a\left(L - x - \frac{L^2 - x^2}{2L}\right) = \frac{t_a L}{2}\left(1-\frac{x}{L}\right)^2 \tag{5a.3.16}$$

At this point, we can see that Eq. 5a.3.16 is the same as Eq. 5a.3.6, so the rest of solution for this case, when we take the section on the left of the cut, reduces to the one described in option A.

Exercise 5a.4 – Torsion with varied shapes

Part 5a.4.1: Cylindrical bar in torsion

Consider the following **shallow** cylinder represented in Figure 5a.4.1. It is made of copper ($G_{Cu} = 45$ GPa). We clamp it and we apply a torque ($T_0 = 5$ kN · m) at the free end. Dimensions are given on Figure 5a.4.1 ($r_1 = 10$ cm, $r_2 = 5$ cm, $L = 5$ m).

a) Draw the free body diagram, write the equilibrium equation, and give the expression of the internal Torque as a function of T_0.

b) Express the polar moment of inertia I_p as a function of r_1, r_2, and L.

c) Determine the minimum shear stress at the section represented in Figure 5a.4.1(a) (dashed red)?

d) Determine the torsion angle at the free end, ϕ_1, as a function of r_1, r_2, L, G_{Cu}, and T_0.

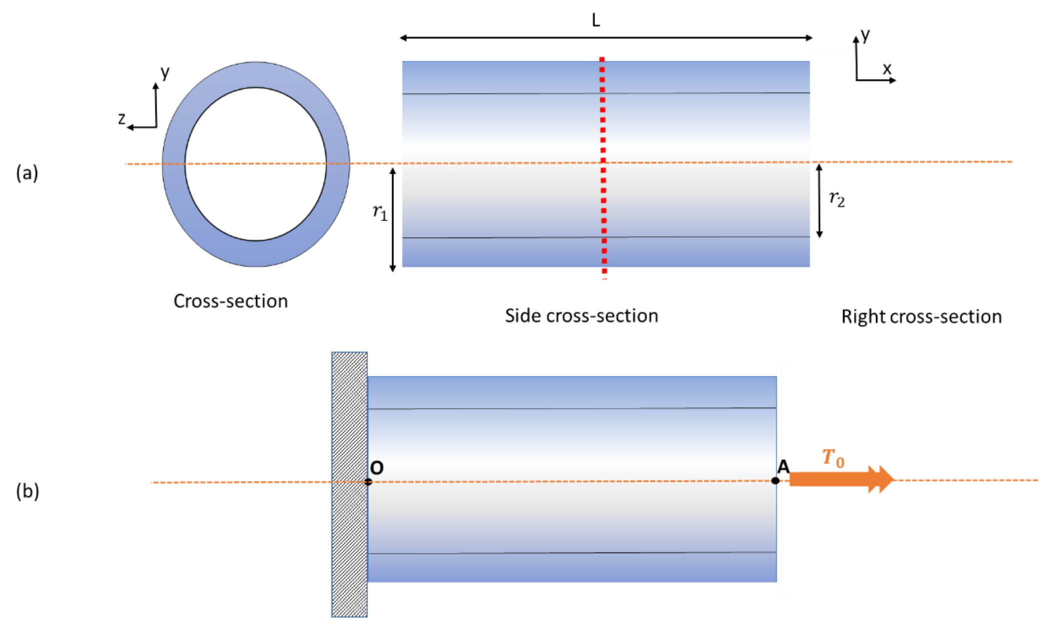
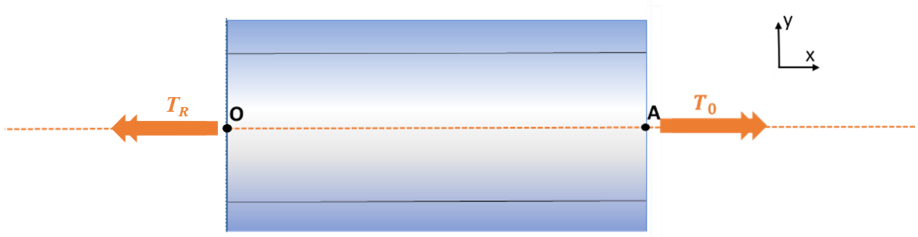


Figure 5a.4.1 | System Description

Solution – 5a.4.1

a) Draw the free body diagram, write the equilibrium equation, and give the expression of the internal Torque as a function of T_0.

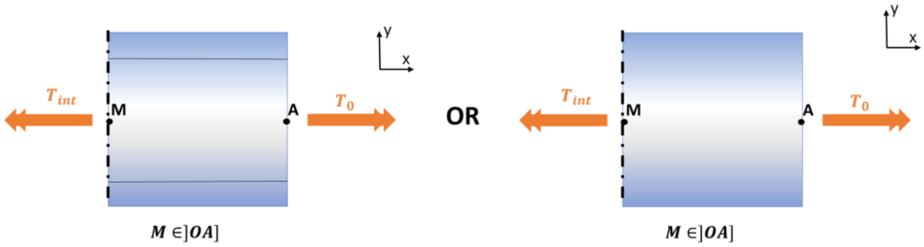
We can draw the free body diagram as:



With

$$-T_R + T_0 = 0 \tag{5a.4.1}$$

We cut the bar between O and A and we call M the point on the cut at the position of the axis. T_{int} is the internal torque at this section.



The equilibrium equations give us :

$$T_0\overrightarrow{e_x} - T_{int}\overrightarrow{e_x} = \vec{0} \quad \text{or} \quad T_0 - T_{int} = 0 \tag{5a.4.2}$$

$$T_0 = T_{int} \tag{5a.4.3}$$

b) Express the polar moment of inertia I_p as a function of r_1 *and* r_2.

From the formula of the polar moment of inertia, we get the following. Inferior limit of integration is the cylinder internal radius.

$$I_p = \int_A r^2 \mathrm{dA} = 2\pi \int_{r_2}^{r_1} r^3 \mathrm{d\rho} = \left[\frac{\pi r^4}{2}\right]_{r_2 > r_1} = \left[\frac{\pi(r_1^4 - r_2^4)}{2}\right] \tag{5a.4.4}$$

c) Determine the minimum shear stress at the section represented in Figure 5a.4.1(a) (dashed red)?

We are in the case of a thick-walled cylinder. Thin wall approximations do not work here. We compute the minimal stress at radius r_2, where the stress is minimal. From there, we get:

$$\tau_{min} = \frac{T_0 r_{min}}{I_p} = 2\frac{T_0 r_2}{\pi(r_1^4 - r_2^4)} = 1.7 \text{ MPa} \tag{5a.4.5}$$

d) Determine the torsion angle at the free end, ϕ_1, as a function of r_1, r_2, L, G_{Cu}, and T_0.

We simply compute the formula of the torsion angle and integrate along the bar. The cross section being uniform, the integral is linear:

$$\phi_1 = \int_0^L \frac{T_0}{I_p G_{Cu}} \mathrm{dx} = \frac{T_0}{I_p G_{Cu}} \int_0^L \mathrm{dx} = \frac{T_0 L}{I_p G_{Cu}} = \frac{2T_0}{\pi G(r_1^4 - r_2^4)} L \tag{5a.4.6}$$

Part 5a.4.2: Conical bar in torsion

Consider the **full** conical cylinder of Figure 5a.4.2. It is made of aluminum ($G_{Al} = 26$ GPa). We clamp it and we apply a torque ($T_0 = 5$ kN · m) at the free end. Dimensions are given on Figure 5a.4.2 ($r_1 = 10$ cm, $r_3 = 5$ cm, $L/2 = 2.5$ m).

- **a) Express the radius of the cylinder as a function of r_1, r_3, L, and x (x is the varying parameter along the x-axis considering O' as the origin).**
- **b) Express the polar moment of inertia, I_p, as a function of r_1, r_3, L, and x.**
- **c) Determine the torsion angle ϕ_2, where ϕ_2 is the torsion angle at the free-end of the aluminum bar as a function of r_1, r_3, L, G_{Al}, and T_0.**

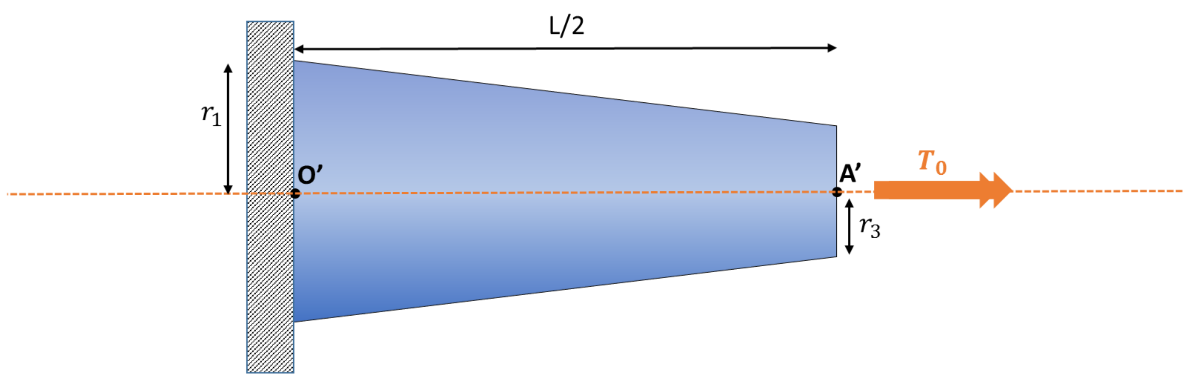


Figure 5a.4.2 | System Description

Solution – 5a.4.2

a) Express the radius of the cylinder as a function of r_1, r_3, L, and x (x is the varying parameter along the x-axis considering O' as the origin).

If we consider the origin at O', then finding the radius is equivalent to finding the coefficients of a linear function in the (x,y) Cartesian coordinate system.

$$r(x) = r_1 + 2\frac{r_3 - r_1}{L}x \quad (5a.4.7)$$

b) Express the polar moment of inertia, I_p, as a function of r_1, r_3, L, and x.

We compute the polar moment of inertia. Integration is done considering only a circular cross-section. Computation of the r(x) formula is done *only after* integration.

$$I_p = \int_A r^2 \mathrm{dA} = 2\pi \int_0^r \rho^3 \mathrm{d}\rho = \left[\frac{\pi\rho^4}{2}\right]_{0>r} = \frac{\pi r^4}{2} = \frac{\pi}{2}\left(r_1 + 2\frac{r_3 - r_1}{L}x\right)^4 \quad (5a.4.8)$$

c) Determine the torsion angle ϕ_2, where ϕ_2 is the torsion angle at the free-end of the aluminum bar as a function of r_1, r_3, L, G_{Al}, and T_0.

The question consists in calculating the integral along the bar.

$$\phi_2 = \int_0^{\frac{L}{2}} \frac{T_0}{I_p(x)G_{Al}} dx = \int_0^{\frac{L}{2}} \frac{T_0}{\frac{\pi}{2}\left(r_1 + 2\frac{r_3 - r_1}{L}x\right)^4 G_{Al}} \mathrm{dx} \quad (5a.4.9)$$

$$= \frac{2T_0}{\pi G_{Al}} \left[\frac{-1}{\frac{6(r_3 - r_1)}{L} \left(r_1 + 2\frac{r_3 - r_1}{L} x \right)^3} \right]_{0 > \frac{L}{2}} = \frac{T_0 L}{3\pi G_{Al}(r_1 - r_3)} \left(\frac{1}{r_3^3} - \frac{1}{r_1^3} \right)$$

Part 5a.4.3: Complex bar in torsion

We clamp the two bars of part 5a.4.1 and part 5a.4.2 together as shown in Figure 5a.4.3. A torque, T_0, is applied at the free end of the bar.

a) Determine the torsion angle, ϕ, at the free end of the bar, as a function of ϕ_1 and ϕ_2.
b) Calculate the numerical value of ϕ.

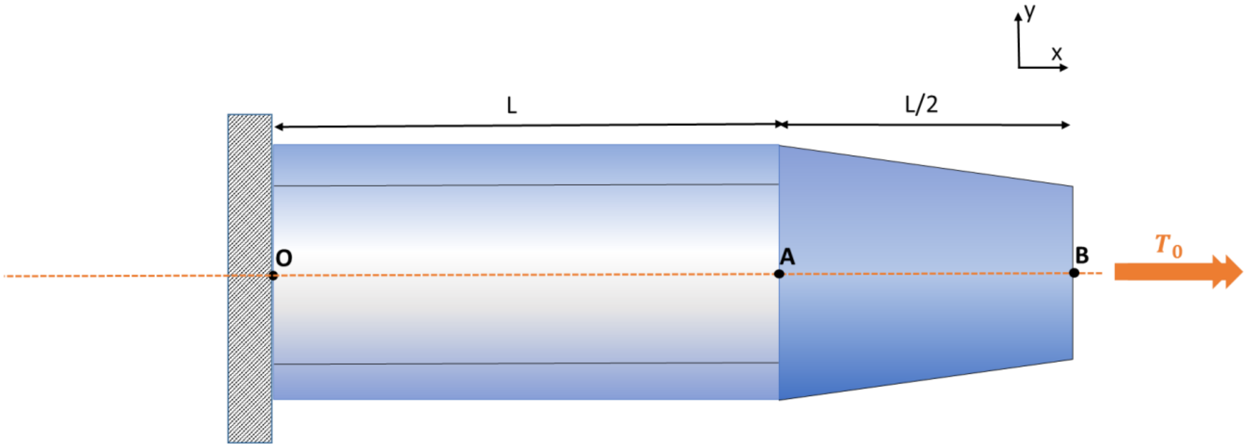


Figure 5a.4.3 | Complex bar with applied torque

Solution – 5a.4.3

a) Determine the torsion angle, ϕ, at the free end of the bar, as a function of ϕ_1 and ϕ_2.

We use the superposition principle:

$$\phi = \phi_1 + \phi_2 \quad (5a.4.10)$$

b) Calculate the numerical value of ϕ.

$$\phi = 3.8 \text{ mrad} + 14.3 \text{ mrad} = 18.1 \text{ mrad} = 1.03° \quad (5a.4.11)$$

Exercise 5a.5 – Torsion in a composite cone

Consider the circular bar in Figure 5a.5 (dimensions: $D = 2$ cm; $L = 96$ cm). The Young's moduli and Poisson's ratios of the bar materials are $E_A = 35$ GPa and $\nu_A = 0.25$ in the left portion and $E_B = 72.8$ GPa and $\nu_B = 0.3$ in the right portion. We apply a torque $T_0 = 10\pi$ N · m.

a) Calculate the numerical value of the shear modulus of both materials, G_A and G_B.
b) Determine the polar moment of inertia, I_P, at position x of the bar as a function of D, x, and L.
c) Calculate the numerical value of the maximum shear stress in the entire bar.
d) Calculate the numerical value of the torsion angle, ϕ, of the free end of the bar.

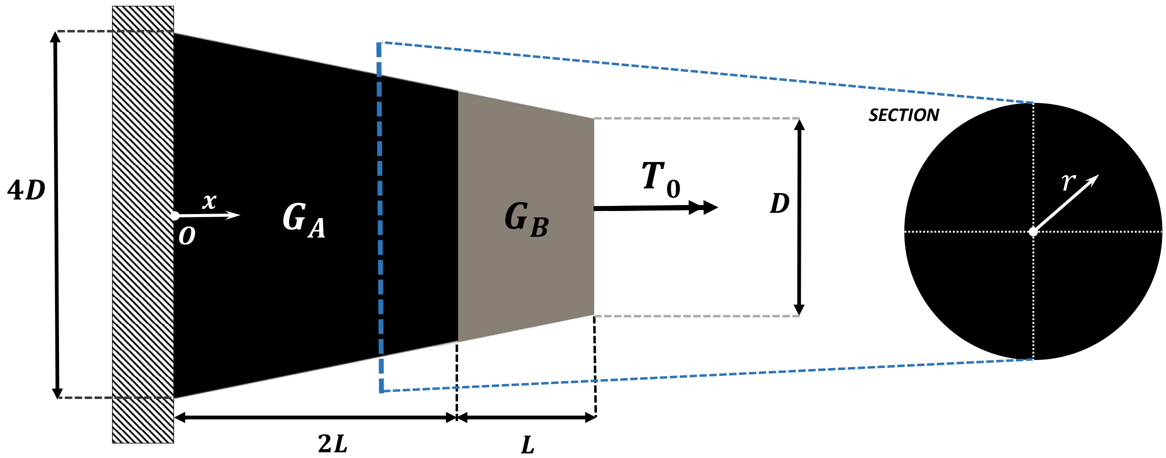


Figure 5a.5 | Schematic of the composite bar in torsion.

Solution – 5a.5

a) Calculate the numerical value of the shear modulus of both materials, G_A and G_B.

$$G_A = \frac{E_A}{2(1+\nu_A)} = \frac{35\text{ GPa}}{2(1+0.25)} = 14\text{ GPa} \tag{5a.5.1}$$

$$G_B = \frac{E_B}{2(1+\nu_B)} = \frac{72.8\text{ GPa}}{2(1+0.3)} = 28\text{ GPa} \tag{5a.5.2}$$

b) Determine the polar moment of inertia (I_P) at position x of the bar as a function of D, x, and L.

$$d(x) = 4D - \frac{D}{L}x \quad \text{OR} \quad r(x) = 2D - \frac{D}{2L}x \tag{5a.5.3}$$

$$I_P(x) = \pi \cdot \frac{d^4}{32} = \frac{\pi}{32}D^4\left(4 - \frac{x}{L}\right)^4 \text{ OR } I_P(x) = \pi \cdot \frac{r^4}{2} = \frac{\pi}{2}D^4\left(2 - \frac{x}{2L}\right)^4 \tag{5a.5.4}$$

c) Calculate the numerical value of the maximum shear stress in the entire bar.

$$\tau(x,r) = \frac{T_0 r}{I_P(x)} \tag{5a.5.5}$$

$$\tau(x,r) = \frac{32T_0\, r}{\pi D^4 \left(4 - \frac{x}{L}\right)^4}$$

$$\tau_{Max}\,(3L, \frac{D}{2}) = \frac{16T_0}{\pi D^3} = \frac{160\pi}{\pi \cdot 8 \cdot 10^{-6}} = 20 \text{ MPa} \quad (5a.5.6)$$

d) Calculate the numerical value of the torsion angle of the free end of the bar.

$$\phi(3L) = \int_0^{3L} \frac{T_0}{G(x) I_P(x)} dx \quad (5a.5.7)$$

$$= \phi(0 > 2L) + \phi(2L > 3L) \quad (5a.5.8)$$

$$= \int_0^{2L} \frac{T_0}{G_A I_P(x)} dx + \int_{2L}^{3L} \frac{T_0}{G_B I_P(x)} dx$$

$$= \frac{T_0}{G_A} \int_0^{2L} \frac{dx}{I_P(x)} + \frac{T_0}{G_B} \int_{2L}^{3L} \frac{dx}{I_P(x)} \quad (5a.5.9)$$

$$= \frac{32T_0}{\pi G_A D^4} \left[\frac{L}{3} \left(\frac{1}{4 - \frac{x}{L}} \right) \right]_0^{2L} + \frac{32T_0}{\pi G_B D^4} \left[\frac{L}{3} \left(\frac{1}{4 - \frac{x}{L}} \right) \right]_{2L}^{3L} \quad (5a.5.10)$$

$$= \frac{T_0}{G_A} * \frac{32}{\pi D^4} \left(\frac{L}{3(4-2)^3} - \frac{L}{3(4)^3} \right) + \frac{T_0}{G_B} * \frac{32}{\pi D^4} \left(\frac{L}{3(4-3)^3} - \frac{L}{3(4-2)^3} \right)$$

$$= \frac{T_0}{G_A} * \frac{32}{\pi D^4} L \left(\frac{7}{192} \right) + \frac{T_0}{G_B} * \frac{32}{\pi D^4} L \left(\frac{56}{192} \right) = \frac{32T_0}{\pi D^4} * \frac{L}{192} \left(\frac{7}{14} + \frac{56}{28} \right) \cdot 10^{-9} \quad (5a.5.11)$$

$$= \frac{32T_0}{\pi D^4} * \frac{L}{192} 2.5 \cdot 10^{-9} \quad (5a.5.12)$$

$$\phi = \frac{32\pi 10}{\pi 16 \cdot 10^{-8}} * \frac{96 \cdot 10^{-2}}{192} 2.5 \cdot 10^{-9} = 25 \text{ mrad} \quad (5a.5.13)$$

(NB: 5 mrad + 20 mrad)